Classification of transformations of probabilities for preparation procedures: trigonometric and hyperbolic behaviours.

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Abstract

We provide frequency probabilistic analysis of perturbations of physical systems by preparation procedures. We obtained the classification of possible probabilistic transformations connecting input and output probabilities that can appear in physical experiments. We found that so called quantum probabilistic rule is just one of possible rules. Besides the well known trigonometric transformation (for example, for the polarization of light), there exist the hyperbolic transformation of probabilities. In fact, 'hyperbolic polarization' have laready been observed in experiments with elementary particles. However, it was not interpreted in such a way. The situation is more complex with the hyperbolic interference of alternatives.

We obtained the classification of possible probabilistic rules that could connect input and output probabilities for preparation procedures (for micro as well as for macro systems), see [1] on the general theory of preparation/measurement procedures. We found that the standard trigonometric probabilistic transformation (so called 'quantum probabilistic rule') is just one of possible rules. Our analysis implies that, besides the trigonometric transformations of probability which appear in the quantum formalism, there

exist hyperbolic transformations of probability. Moreover, such transformations connect input and output probabilities in the well known experiments with elementary particles: trigonometric 'polarization' in some directions and hyperbolic 'polarization' in other directions.

In fact, in this paper we follow to so called contextualist approach to physical probabilities, see, for example, [1], [2] for the details. Probability distributions of physical variables are determined not only by objective properties of physical systems, but also by the whole context of a preparation/measurement procedure, see N. Bohr [3]. We point out that the contextualist framework is characterized by a large diversity of viewpoints. In particular, there are various viewpoints on the origin of quantum stochasticity. It might be that quantum stochasticity can be reduced to classical stochasticity. It might be not. In fact, in the present paper we demonstrate that quantum stochasticity can be reduced to classical stochasticity (at least simulated for macro systems).

This paper is closely related to the paper [4] (which was also presented as a plenary lecture at International Conference "Foundations of Probability and Physics", Vaxjo, Sweden-2000). We hope that this present text gives clearer presentation of our ideas.

1 Classification of probabilistic transformations in Nature

We shall use so called frequency approach to probability which was developed by R. von Mises [5], see also [6]. In that approach the probability is defined as the limit of relative frequencies. [1]

Let S be an ensemble of physical systems. Let $a(= a_1, a_2)$ be a dichotomic physical variable which can be measured for elements of the ensemble S.

[1]Such an approach is quite natural for physicists, see, for example, A. Peres in [1]. However, it was strongly criticized in mathematical literature, see, for example, [6], because there were some problems with the definition of randomness. We would not use the notion of randomness in our considerations. Therefore all this critique has no relation to our considerations. The only thing that we shall use is that the probability is nothing than the limit-value of the corresponding relative frequency. The frequency approach to probability is very useful for probabilistic analysis of physical phenomena. In such a way we can obtain some results that it would be impossible to obtain in the conventional approach to probability that is based on the abstract Kolmogorov axiomatics, [7].

Probabilities of values $a_i, i = 1, 2$, are defined as

$\mathbf{p}_i = \lim_{N\to\infty} \mathbf{p}_i(N), \;\; \mathbf{p}_i(N) = \frac{n_i}{N}$.

Here $N = |S|$ and $n_i = |\{s \in S : a = a_i\}|$, where the symbol $|O|$ denotes the number of elements in the ensemble O.

Let $\mathcal{E}$ be some preparation procedure; see [1] (see also P. Dirac [8]: "In practice the conditions could be imposed by a suitable preparation of the system, consisting perhaps in passing it through various kinds of sorting apparatus, such as slits and polarimeters, the system being left undisturbed after the preparation."). Suppose that by applying $\mathcal{E}$ to the ensemble S we produce a new ensemble S'. It is assumed that $\mathcal{E}$ does not change the size of population:

$N = |S| = |S'|$.

All our considerations are based on the simple remark that a preparation procedure may change the probability distribution of values of a. Probabilities of $a_i, i = 1, 2$, after the preparation procedure $\mathcal{E}$ (output probabilities) are defined as

$\mathbf{p}'_i = \lim_{N\to\infty} \mathbf{p}'_i(N), \;\; \mathbf{p}'_i(N) = \frac{n'_i}{N}, \; i = 1, 2,$

where $n'_i = |\{s \in S' : a = a_i\}|$. We make the following trivial calculations:

$\mathbf{p}'_i(N) = \frac{n'_i}{N} = \frac{n_i}{N} + \delta_i(N)$.

Here

$\delta_i(N) = \frac{n'_i - n_i}{N} = \mathbf{p}_i(N)\lambda_i(N)$,

where

$\lambda_i(N) = \frac{n'_i - n_i}{n_i}$.

Finally, we get

$$\mathbf{p}'_i(N) = \mathbf{p}_i(N)(1 + \lambda_i(N)). \tag{1}$$

The coefficient $\lambda_i(N)$ gives the statistical deviation for the distribution of a induced by the preparation procedure $\mathcal{E}$. We note that there exist limits

$\lambda_i = \lim_{N\to\infty} \lambda_i(N)$.

This is a consequence of the existence of limits for relative frequencies $\mathbf{p}_i(N)$ and $\mathbf{p}'_i(N)$.[2] Thus by taking the limit when $N \to \infty$ in (1) we get the following relation between the input and output probabilities:

$$\mathbf{p}'_i = \mathbf{p}_i(1 + \lambda_i). \tag{2}$$

[2] In fact, we (as always in modern physics) assume that the preparation procedure $\mathcal{E}$ is statistically regular: all relative frequencies stabilize when $N \to \infty$, compare with [6] where we considered the possibility that the law of the statistical stabilization (the law of large numbers) might be violated for some physical variables.

This is the general probabilistic transformation which could be produced by natural (or even social) statistical phenomena. We remark that the coefficients λ_1 and λ_2 are not independent. There is the normalization condition:

$1 = \mathbf{p}'_1 + \mathbf{p}'_2 = \mathbf{p}_1 + \mathbf{p}_2 + \lambda_1 \mathbf{p}_1 + \lambda_2 \mathbf{p}_2$.

We obtained a kind of orthogonality relation:

$$\lambda_1 \mathbf{p}_1 + \lambda_2 \mathbf{p}_2 = 0. \tag{3}$$

Magnitudes of the coefficients λ_i will play the crucial role in our analysis.

(T) Let $|\lambda_i| \leq 1, i = 1, 2$. This means that the relative magnitude of perturbations of frequencies is not so large. Here

$$|n'_i - n_i| \leq n_i, N \to \infty. \tag{4}$$

Thus perturbations induced by the $\mathcal{E}$ may change statistics strongly, but not crucially, compare with the case **(HT).** The coefficients λ_i can be represented in the form

$\lambda_1 = \cos\theta_1, \quad \lambda_2 = \cos\theta_2$,

where θ_1 and θ_2 are some 'phases.' Orthogonality relation (3) implies that phases θ_1 and θ_2 are not independent:

$$\cos\theta_1 \mathbf{p}_1 + \cos\theta_2 \mathbf{p}_2 = 0. \tag{5}$$

Thus

$$\frac{\cos\theta_2}{\cos\theta_1} = -\frac{\mathbf{p}_1}{\mathbf{p}_2}. \tag{6}$$

Thus if magnitude of the statistical deviation induced by a preparation procedure is not so large, see (4), then we obtain the following rule for the transformation between input and output probabilities:

$$\mathbf{p}'_1 = \mathbf{p}_1(1 + \cos\theta_1) = 2\mathbf{p}_1 \cos^2 \frac{\theta_1}{2}\ , \tag{7}$$

$$\mathbf{p}'_2 = \mathbf{p}_2(1 + \cos\theta_2) = 2\mathbf{p}_2 \cos^2 \frac{\theta_2}{2}\ . \tag{8}$$

We call transformation (7), (8) the *trigonometric probabilistic rule.* In particular, if $\mathbf{p}_1 = \mathbf{p}_2 = 1/2$, we get (on the basis of (7), (8) and (5)) the probabilistic rule for *light polarization*:

$\mathbf{p}'_1 = \cos^2\ \alpha, \ \ \mathbf{p}'_2 = \sin^2\ \alpha$,

where $2\alpha = \theta_1 = \theta_2 + \pi$. We now consider an important particular case of T-probabilistic behaviour.

(C) Let a preparation procedure $\mathcal{E}$ produce negligibly small statistical deviation:

$\lim_{N\to\infty} \lambda_i(N) = \lim_{N\to\infty} \frac{n_i'-n_i}{n_i} = 0, i = 1,2.$

In such a case we have $\mathbf{p}_i' = \mathbf{p}_i, i = 1,2$. This is so called classical probabilistic behaviour (compare with P. Dirac [8], p. 11).

T-probabilistic behaviour (and, in particular, classical and quantum behaviours) has been already observed in various physical experiments. We are going to consider new probabilistic behaviours which have not been yet observed in physical experiments. However, in our frequency considerations those new probabilistic behaviours are not less natural than T-behaviour.

We remark that the coefficients λ_1 and λ_2 have opposite signs. We can assume that $\lambda_1 \geq 0$ and $\lambda_2 \leq 0$. As $p_2' \geq 0$, we get that the coefficient λ_2 must always belong to the interval $[-1, 0]$. Thus this coefficient can be always represented in the form:

$\lambda_2 = \cos\ \theta_2.$

On the other hand, λ_1 can be less as well as larger than 1. In the first case we can represent it as $\lambda_1 = \cos\ \theta_1$; in the second case $\lambda_1 = \cosh\ \ \theta_1$. The first case has been already studied, see (**T**). We now study the second case.

(HT) Let $\lambda_1 > 1$ and $-1 \leq \lambda_2 \leq 0$. This means that the statistical deviation for $a = a_1$ is sufficiently large:

$|n_1' - n_1| > n_1,\ N \to \infty.$

Thus perturbations induced by the $\mathcal{E}$ change crucially the statistics of $a = a_1$. On the other hand, the statistical deviation for $a = a_2$ is relatively small:

$|n_2' - n_2| \leq n_2,\ N \to \infty.$

Thus perturbations induced by the $\mathcal{E}$ change slightly statistics of $a = a_2$. Orthogonality relation (3) implies that phases θ_1 and θ_2 are not independent:

$$\cosh\ \theta_1 \mathbf{p}_1 + \cos\ \theta_2 \mathbf{p}_2 = 0. \tag{9}$$

Thus

$$\frac{\cos\ \theta_2}{\cosh\ \theta_1} = -\frac{\mathbf{p}_1}{\mathbf{p}_2}. \tag{10}$$

We get the *hyperbolic/trigonometric* probabilistic rule:

$$\mathbf{p}_1' = \mathbf{p}_1(1 + \cosh\ \theta_1) = 2\mathbf{p}_1 \cosh^2 \frac{\theta_1}{2}\ , \tag{11}$$

$$\mathbf{p}_2' = \mathbf{p}_2(1 + \cos\ \theta_2) = 2\mathbf{p}_2 \cos^2 \frac{\theta_2}{2}\ . \tag{12}$$

Example. Let $\mathbf{p}_1 = 1/4, \mathbf{p}_2 = 3/4$ and let $\mathbf{p}_1' = a, \mathbf{p}_2' = 1 - a$, where $a \in (1/2, 1]$. We cannot represent $\mathbf{p}_1' = 2\mathbf{p}_1 \cos^2\theta$ for any θ. Here we need to use the representation: $\mathbf{p}_1' = 2\mathbf{p}_1 \cosh^2\theta$, where θ changes between 0 and arcosh $\sqrt{2}$ for a changing between 1/2 and 1. Thus there is no usual trigonometric wave. There is a kind of hyperbolic wave.

Remark. In fact, the use of the $\cos\theta$ (and $\cosh\theta$) representations of the coefficients λ_i is motivated by the quantum formalism (and even the classical field theory, compare with P. Dirac [8]). In principle we can represent $|\lambda| \leq 1$ as $\lambda = f(\theta)$ where f is any function $|f| \leq 1$. Dependence $f(\theta)$ is related to the dependence of the preparation procedure on some parameter $\mathcal{E} = \mathcal{E}(\theta)$. In many experiments f is determined by the space geometry of the experiment. For example, in the experiments with the light polarization we use the Euclidean geometry to modify the preparation procedure $\mathcal{E} = \mathcal{E}(\theta)$. This induces the cos-factor. It is possible to construct families of preparation procedures $\mathcal{E}(\theta)$ connected with other functions $f(\theta)$.

2 Experiments of 'hyperbolic polarization'

We note that both types of probablistic transformations (purely trigonometric and hyper-trigonometric) do happen in quantum experiments and are routinely observed. If we have prepared an ensemble of dichotomic systems such that they exhibit property a_1 with probability $\mathbf{p}_1$ and property a_2 with probability $\mathbf{p}_2$ (with $\mathbf{p}_1 + \mathbf{p}_2 = 1$), then you can always change parameters of the preparation procedure in such a way that input probablities $\mathbf{p}_1, \mathbf{p}_2$ will be transformed into output probablities $\mathbf{p}_1', \mathbf{p}_2'$, where $\mathbf{p}_1', \mathbf{p}_2'$ can have ANY values between 0 and 1 (and of course, we again have $\mathbf{p}_1' + \mathbf{p}_2' = 1$).

In experiments with polarized neutrons such experiments are often done. We can prepare an arbitrary spin state, whose projection in a Stern-Gerlach apparatus gives you the two possible outcomes with probabilities $\mathbf{p}_1, \mathbf{p}_2$. But before the projection you can let the spin pass through a magnetic field around which it precesses, and then you can adjust ANY DESIRED $\mathbf{p}_1', \mathbf{p}_2'$.

3 Physical consequences

1). We demonstrated that there is nothing mysterious in so called the 'quantum probabilistic rule', compare with P. Dirac [8], R. Feynman [9] (see also [10]). This rule can be derived by taking into account perturbation effects of preparation/measurement procedures.[3]

2). We need not apply to 'wave features' of quantum systems. The model can be purely corpuscular. It seems to be that quantum waves are just probablistic waves (induced by statistical perturbations of preparation procedures).

3). The transition from classical behaviour to quantum behaviour is the transition from experiments with negligibly small statistical deviations to experiments with nontrivial statistical deviations, compare with P. Dirac [8].

4). Our analysis did not demonstrate any difference in probabilistic behaviour of macro and micro systems. 'Quantum probabilistic behaviour' might be observed in experiments with macro systems, 'polarization' and 'interference' of macro balls.

Our investigation was induced by investigations of J. Summhammer [11] on the origin of the quantum probablistic rule. I would like to thank J. Summhammer for numerous (critical) discussions.

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[3] In fact, P. Dirac pointed out that such effects play the crucial role in the creation of quantum behaviour [8]. However, he did not pay attention to the possibility to find the purely probabilistic explanation of the origin of 'quantum probabilistic rule.' He must use wave arguments [8]: "If the two components are now made to interfere, we should require a photon in one component to be able to interfere with one in the other." The same was done by N. Bohr [5] who had to introduce the principle of complementarity to combine corpuscular and wave properties of elementary particles.

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